Synergistic Motifs in Gaussian Systems

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MOTIVATION

Higher-order interdependencies are central features of complex systems, **yet a mechanistic explanation for their emergence remains elusive.** For linear Gaussian systems of arbitrary dimension, we derive an expression for synergy-dominance in terms of signed network motifs in the system's correlation matrix. We prove that: (i) antibalanced correlational structures ensure synergy-dominance (i.e., negative O-information) (ii) antibalanced triads in the dyadic interaction matrix of Ornstein–Uhlenbeck processes are necessary for synergy-dominance. Our theoretical results thus demonstrate that pairwise interactions alone can give rise to synergistic information in the absence of explicit higher-order mechanisms. We provide **empirical evidence for the analytical link between information theory (IT) and structural balance theory (SBT)** by performing an analysis similar to that described in Saberi et al (2024) using fMRI data recorded during cognitive tasks. Our results **highlight SBT as an instrumental conceptual framework to study higher-order interdependencies**.

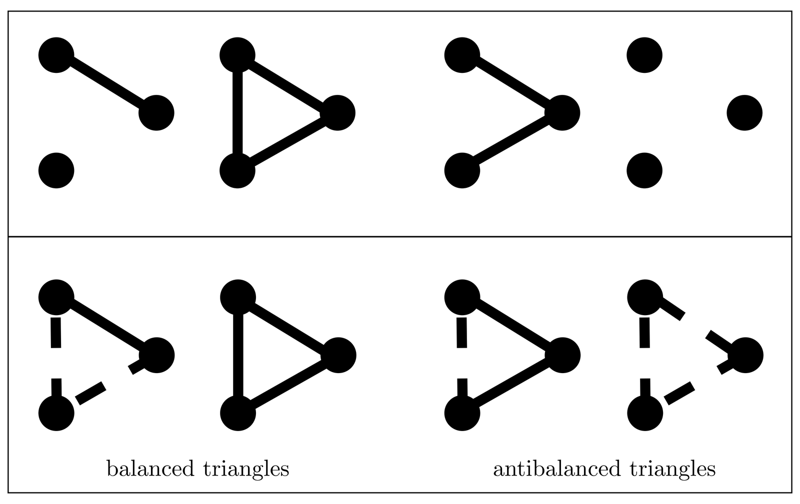


Figure 1: (Top) All the non-isomorphic undirected unweighted graphs of size $N = 3$. **(Bottom)** All the non-isomorphic complete signed graphs of size $N = 3$. Solid (dashed) edges indicate **positive** (**negative**) connections or correlations. Antibalanced triangles are the building blocks of all antibalanced complete signed graphs of higher dimension (e.g., see **insets** in **Fig. 2 bottom**). We use the set of all non-isomorphic complete signed graphs to represent all the possible correlational structures of a system of size N.

Theoretical Results

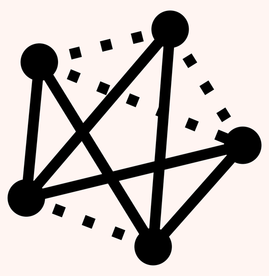
Static Systems Results The O-information for Gaussian systems $\mathbf{X}$, defined as $\Omega(\mathbf{X}) = \sum_{i=1}^{N} TC(\mathbf{X}_{-i}) - (N-2)TC(\mathbf{X})$, can be written as

$$\Omega(\mathbf{X}) = \frac{N-2}{2}\log\left[\det(\Sigma)\right] - \frac{1}{2}\sum_{i}\log\left[\det(\Sigma_{-i})\right]. \tag{1}$$

This expression is negative for synergy-dominated systems and positive for redundancy dominated systems. By expanding the log determinants using $\log\left[\det(\Sigma)\right] = -\sum_{k=2}\frac{(-1)^k}{k}\mathrm{Tr}[W^k]$, where $W = \Sigma - \mathbb{I}$, we obtain

$$\boxed{\Omega(\mathbf{X}) = \frac{1}{2}\sum_{k=3}^{\infty}\frac{1}{k}\Big[(-1)^{k-1}\sum_{w\in\boldsymbol{w}^k}(|w|-2)\,d(w)\Big]} \tag{2}$$

where $w = (i_1, i_2, \ldots, i_k, i_1) \in \boldsymbol{w}^k$ is a closed walk from the set of all closed walks of length k, $d(w) = \prod_{i=1}^{k} W_{w_i,w_{i+1}}$ denotes the weight of a walk, and $|w|$ denotes the number of distinct nodes in the closed walk. **Eq. (2)** renders explicit the link between the O-information and the motifs in the correlational matrix of a Gaussian system. Namely, even-length closed walks with $d(w) > 0$ and odd-length closed walks with $d(w) < 0$ contribute synergistically



As per the ***structure theorem for antibalance***, a graph G is antibalanced if it can be partitioned into two non-overlapping sets of vertices, V_1 and V_2 (at least one non-empty), such that edges between vertices in the same set are negative and edges between nodes in distinct sets are positive.

Thus, if W is the adjacency matrix of an antibalanced graph, $d(w) > 0$ when w is a walk of even length, and $d(w) < 0$ for odd-length walks, ensuring **Eq. (2)** to be negative.

Dynamical Systems Results We consider continuous Ornstein-Ulenbech (OU) processes

$$d\mathbf{X}(t) = -\mathbf{X}(t)\cdot(\mathbb{I} - A)dt + d\boldsymbol{\mathcal{W}}(t) \tag{3}$$

where A is the interaction matrix and $\boldsymbol{\mathcal{W}}$ is a Wiener process with covariance matrix equal to $\mathbb{I}$. If A is symmetric and Schur stable, the covariance matrix $\Sigma^\star$ of the resulting dynamics can be written as $\Sigma^\star = \frac{1}{2}(\mathbb{I} - A)^{-1}$. Using this expression, we show for systems of size $N = 3$ that $\Omega(\mathbf{X}) < 0$ if

$$-2xyz > (xy)^2 + (xz)^2 + (yz)^2 - (xyz)^2. \tag{4}$$

where $x = A_{12}, y = A_{13}, z = A_{23}$. Since $|x|, |y|, |z| < 1$, we require the interaction matrix to be antibalanced (i.e., $xyz < 1$) for OU processes of size 3 to be synergy-dominated (Fig 2, top).

Next, we asked whether antibalanced interaction structures promote the emergence of synergistic information in larger dynamical systems (Fig 2, bottom).

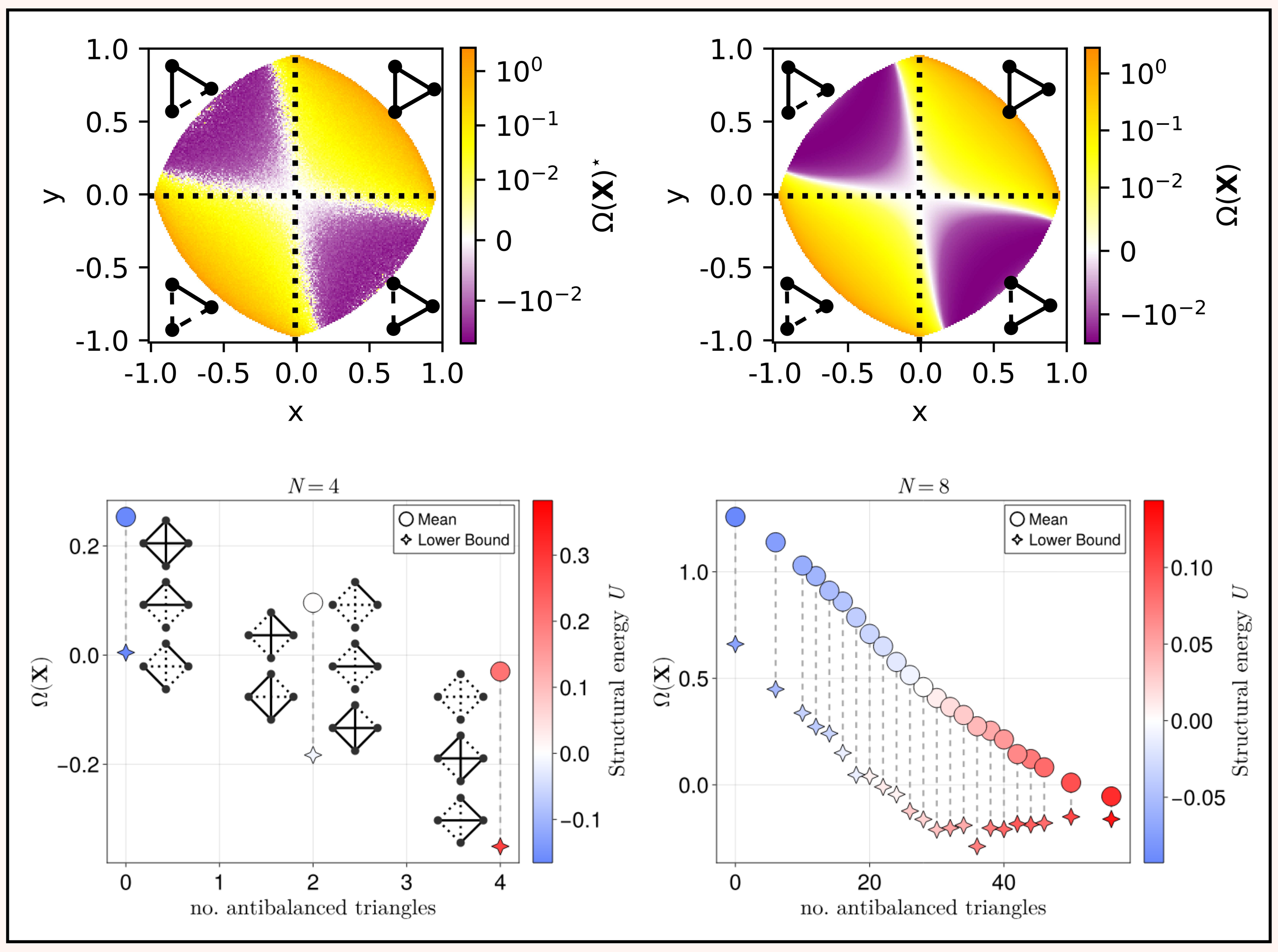


Figure 2: (Top) Numerical $\Omega(\boldsymbol{X})^\star$ (left) and theoretical $\Omega(\boldsymbol{X})$ (right) results for 3-dimensional Ornstein–Uhlenbeck (OU) processes with varying interaction matrix elements $A_{12} = x, A_{13} = y$ and fixed $A_{23} = z = 0.25$. Insets depict the unweighted complete signed graphs, with adjacency matrix sign(A), corresponding to each quadrant (unlabeled since $\Omega(\boldsymbol{X})$ is order invariant).

(Bottom) Mean $\Omega(\boldsymbol{X})$ as a function of the number of antibalanced triangles in the interaction matrix A. The lower bound indicates the lowest $\Omega(\mathbf{X})$ encountered in our numerical exploration across all configurations with a specific no. of antibalanced triangles. Each data point for the mean $\Omega(\mathbf{X})$ is coloured according to its corresponding mean structural energy value, $U = \binom{N}{3}^{-1}\sum_{w\in\boldsymbol{w}^3}\sqrt[3]{d(w)}$, interpreted here as a measure of (anti)balance. Each interaction matrix A was generated with spectral radius $\rho(A) = 0.9$ for all possible non-isomorphic complete signed graphs of size N.

Empirical Validation

We provide empirical evidence (**Fig. 3-5**) for the relationship between the structural energy $U = \binom{N}{3}^{-1}\sum_{w\in\boldsymbol{w}^3}\sqrt[3]{d(w)}$, and the O-information Ω using the dataset from Saberi et al., 2024.

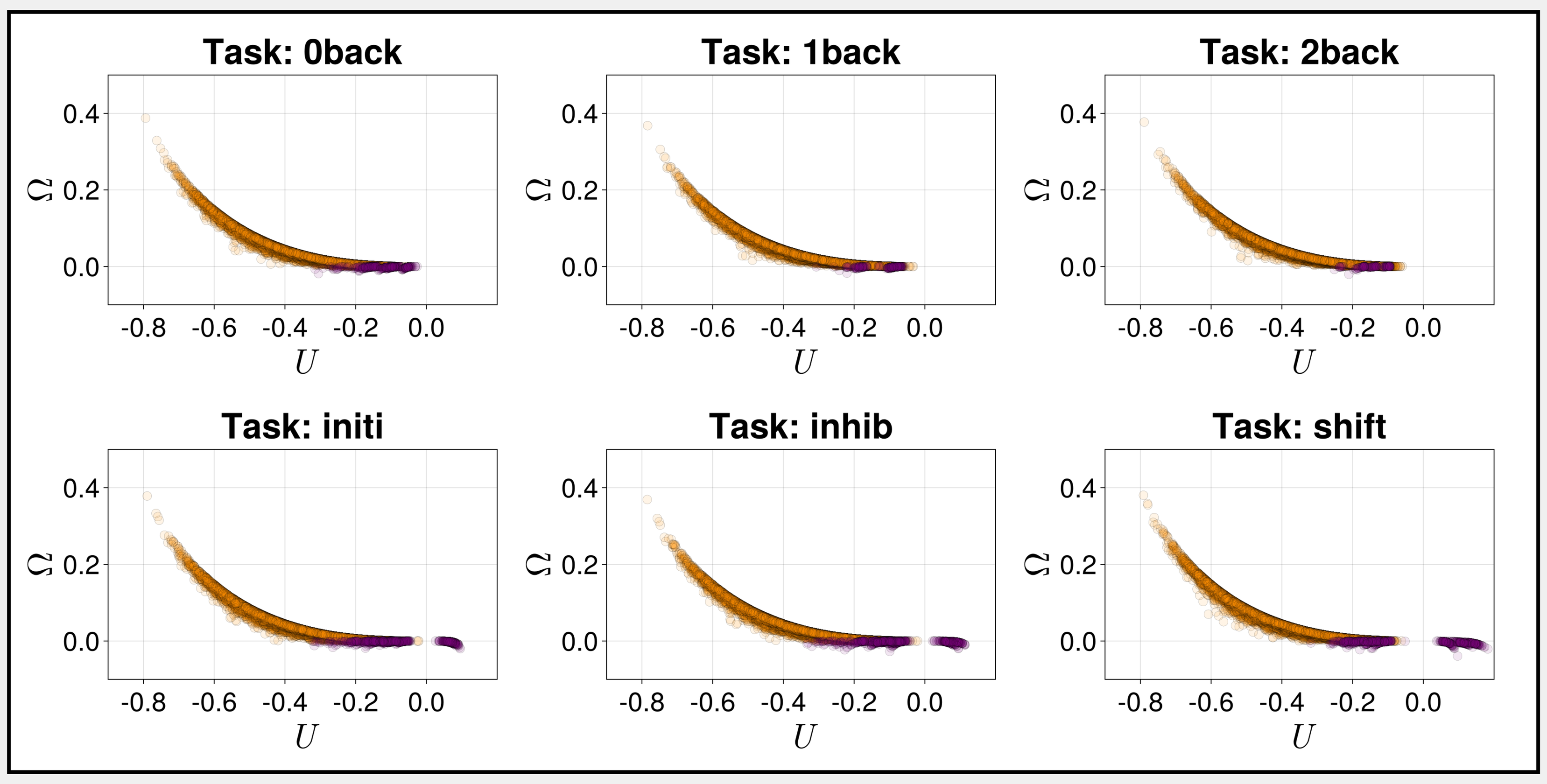


Figure 3 Ω as a function of U for **each triad** in the correlation matrix of the **whole brain** across cognitive tasks. Synergistic (redundant) triplets highlighted in purple (orange).

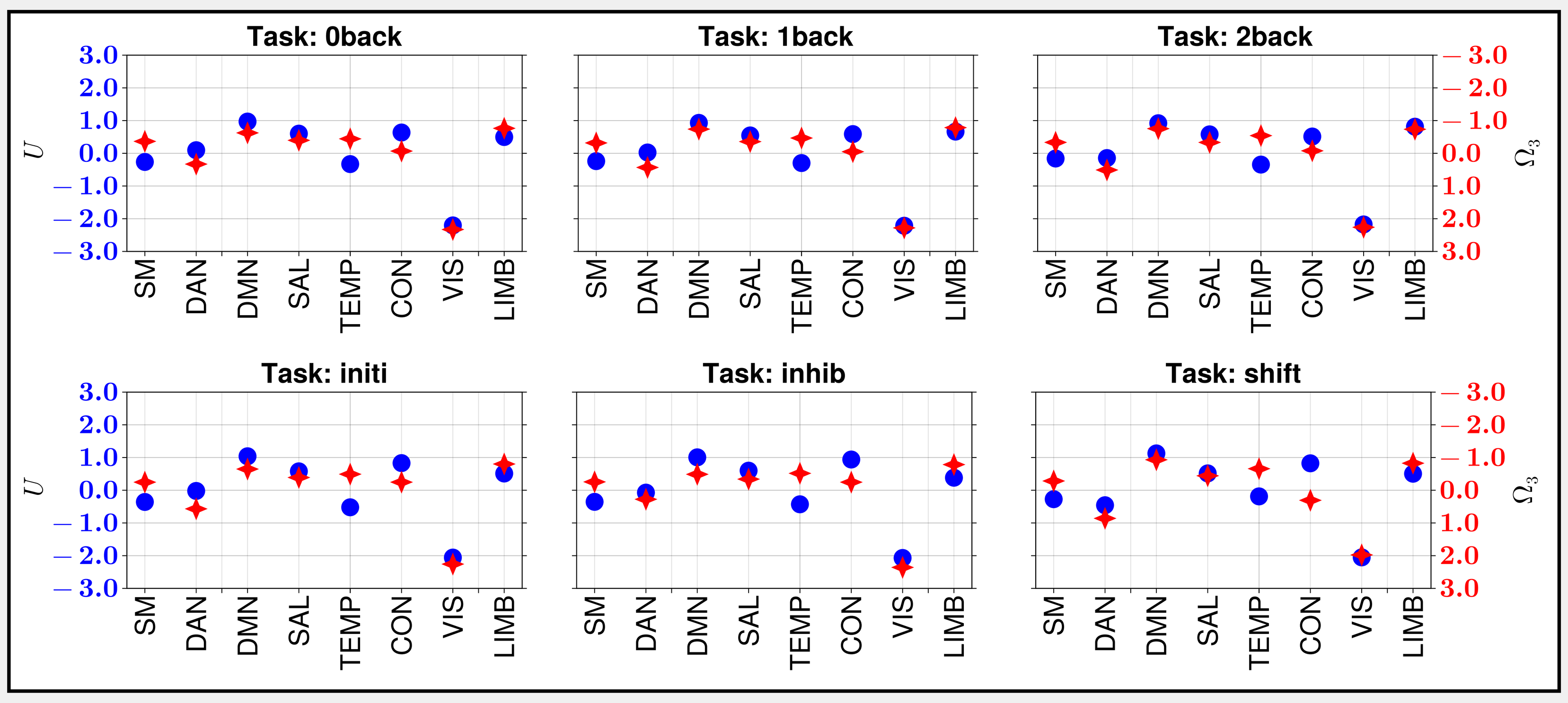


Figure 4 Z-scored group-level mean U and mean O-information across triads (Ω_3) for each **canonical functional network** across cognitive tasks. Ω_3 is computed as the average O-information across all triads within the same functional network.

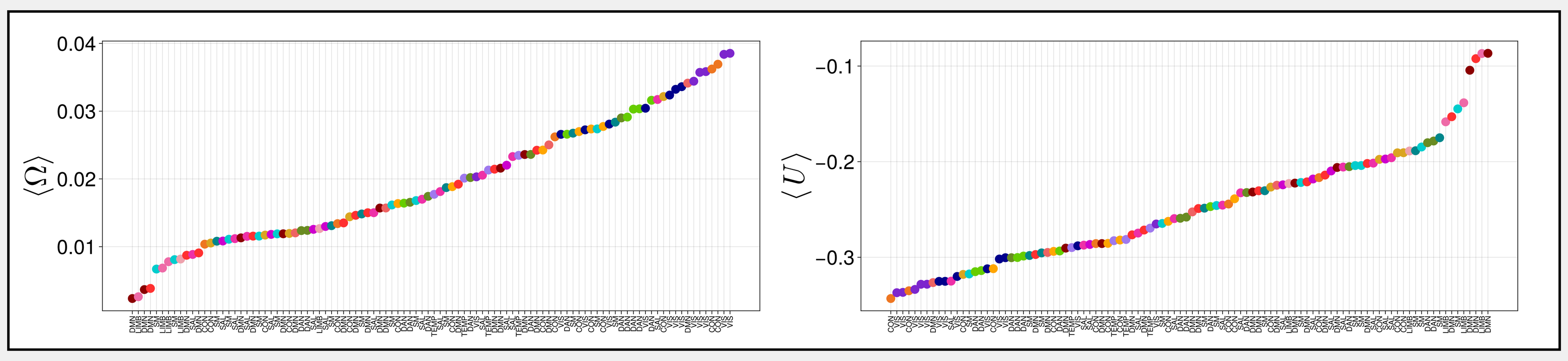


Figure 5 **Single-node** mean contributions of Ω (left) and U (right). Metrics were computed as the average Ω and U across all triads in which each node participates. Results are shown for the shifting task; each node is labeled according to the canonical network to which it belongs.

CONCLUSIONS

We proved that antibalanced pairwise interactions are necessary for synergy-dominance in linear Gaussian systems without higher-order mechanisms (**Fig. 2**), and that antibalanced correlational structures ensure synergy-dominance. In doing so, we reveal SBT as an instrumental conceptual lens for studying higher-order interdependencies. We provide evidence of our analytically informed interpretation of the relation between SBT and IT by performing a similar analysis as in Saberi et al., 2024. We show that the structural energy and the O-information provide similar information about the system at three levels: whole network (**Fig. 3**), functional subnetwork (**Fig. 4**) and node levels (**Fig. 5**).

Finally, it will be interesting to uncover the network-level dynamics—distinct from dynamics on networks with fixed topology—that lead to the formation of antibalanced interaction structures. In social balance theory, the formation of balanced triangles has largely been linked to higher-order mechanisms (e.g., triadic interactions), with only recent work showing that a simple pairwise homophily-based mechanism suffices Pham et al., 2022. Whether similar network mechanisms exist for the antibalanced case is an interesting and unexplored research question.

References

Pham, T. M., Korbel, J., Hanel, R., & Thurner, S. (2022).Empirical social triad statistics can be explained with dyadic homophylic interactions. *Proceedings of the National Academy of Sciences*, *119*(6). https://doi.org/10.1073/pnas.2121103119

Saberi, M., Rieck, J. R., Golafshan, S., Grady, C. L., Misic, B., Dunkley, B. T., & Khatibi, A. (2024).The brain selectively allocates energy to functional brain networks under cognitive control. *Scientific Reports*, *14*(1). https://doi.org/10.1038/s41598-024-83696-7

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